ON A CLASS OF REAL BANACH SPACES

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ABSTRACT

The structure theory for simplex spaces is extended to arbitrary real Banach spaces with L^1–duals.

1. Introduction

It has long been known that the structure of an operator algebra and its representations is reflected in the convex structure of its state space. In the past five years this has led to new methods for studying convex sets (see [3, 23]). This progress was in part inspired by Kadison's early investigation of "function systems" [14]. These may be used to realize any compact convex subset of a Hausdorff locally convex space as the "state space" of a certain ordered Banach space. As would be expected, the most detailed theory has been developed for the simplest class of convex sets, the Choquet simplexes (see [10, 11, 12, 13, 22]). Recently Lazar and Lindenstrauss [17, Th. 2.1] have shown that one of the basic results of the latter theory, the Edwards Extension Theorem, generalizes to certain, *non-ordered* Banach spaces, which we shall call "Lindenstrauss spaces". These are the Banach spaces with dual an L^1-space. In this context the unit ball of the dual plays the role of the state space. We shall show that almost the entire existing structure theory of Choquet simplexes carries over to the Lindenstrauss spaces.

At several points in this development it would have been convenient to appeal to the Cartier Meyer Dilation Theorem [20, p. 232]. Since the non-separable form of this result is known only for simplexes, it has been necessary to prove a new result in this direction. In Theorem 2.1 it is shown that the Cartier Lemma (see [21, Prop. 13.1]) may be strengthened to allow approximation of maximal dilations of measures by *maximal* dilations of point masses.

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The maximal Cartier Lemma is first used to give a new solution of a boundary value problem for a compact convex K. Specifically, a characterization is given in Theorem 2.3 for those functions on the closure of the extreme points of K that extend to continuous affine functions on K. This differs from the Alfsen criteria [2] and those of Boboc and Bucur [6] in that it avoids the use of envelopes or non-extremal measures.

Next we consider the representation of a Banach space V as functions on Z, the closure of the extreme points $E(K)$ of the dual ball K. Due to Lazar's recent measure-theoretic characterization of Lindenstrauss spaces [16], such spaces have a particularly simple representation as all of the odd 'affine' functions on Z (Corollary 3.3).

In §4 we show that if V is a Lindenstrauss space, then one may introduce a "structure topology" on $E(K)$. After an identification of antipodal points, this may be regarded as the analogue of the primitive ideal space of a C^*-algebra or of the maximal ideal space of a simplex. A closed set in this topology is just the extreme points of a closed "biface" in K. The key result is that the convex hull of two bifaces is again a biface (Proposition 4.6). This is based on a non-ordered form of the Riesz decomposition property that holds for L-spaces (Lemma 4.4).

In Theorem 5.7 we use the representation theorem for a Lindenstrauss space V on Z to prove that structurally continuous functions on $E(K)$ act as "multipliers" for V. This is a generalization of the Dauns-Hofmann Theorem for C^*-algebras, which as Fell has pointed out may also be proved via a representation theorem for C^*-algebras (see [1, §4]). "Symmetrically dilated" sets are then introduced, and using the maximal Cartier Lemma a second time, it is proved in Theorem 5.8 that their convex hulls are bifaces. This leads to a characterization of the structurally closed sets. The material of § 5 is based on a careful analysis of certain maximal odd measures on K.

In § 6 we show that if V is a separable Lindenstrauss space, then the following properties are equivalent: (1) $E(K)$ has Hausdorff structure (after identification of antipodes), (2) the closure Z of $E(K)$ is contained in the "extremal diameters", and (3) V is a G-space.

Finally in Theorem 7.7 we prove that the Gleit-Taylor Theorem [13; 22] generalizes without change: If V is a separable Lindenstrauss space, then for the structure topology the properties of first countable, second countable, and local compactness are equivalent.

In § 8 we discuss some open problems and directions for further work.

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2. Dilations and boundary value problems

Throughout this paper we shall use the following notation. If X is a compact Hausdorff space, $C(X)$ is the ordered Banach space of real continuous functions on X, with the usual order and uniform norm. $M(X)$ is the ordered Banach dual of $C(X)$, the regular Borel signed measures on X. Both $C(X)$ and $M(X)$ are lattice ordered, and we let $\wedge$, $\vee$, $^+$, $^-$ be the usual lattice operations. We let $P(X)$ be the probability measures on X, i.e., the $\mu \in M(X)$ with $\mu \geqq 0$, $\mu(1) = 1$. We let $\delta_x = \delta(x)$ be the unit mass concentrated at x. If $\mu \in M(X)$, we indicate its support by supp μ.

Let K be a compact convex subset of a locally convex Hausdorff space W. A face Q in K is a convex subset of K such that if $\alpha p + (1-\alpha)q \in Q$, $0 < \alpha < 1$, $p, q \in K$, then $p, q \in Q$. A point p in K is extreme if and only if $\{p\}$ is a face in K. Let $E(K)$ be the extreme points of K, and $Z = Z(K)$ be the closure of $E(K)$. Let $S(K) \subseteq C(K)$ be the wedge of convex functions s in $C(K)$, i.e., those functions s with

$$s(\alpha p + (1-\alpha)q) \leqq \alpha s(p) + (1-\alpha)s(q)$$

$p, q \in K$, $0 \leqq \alpha \leqq 1$. $A(K) = S(K) \cap -S(K)$ consists of the affine continuous functions on K, i.e., those functions $a \in C(K)$ satisfying

$$a(\alpha p + (1-\alpha)q) = \alpha a(p) + (1-\alpha)a(q)$$

$p, q \in K$, $0 \leqq \alpha \leqq 1$. The dilation order on $P(K)$ is defined by $\mu \prec \nu$ if $\mu(s) \leqq \nu(s)$ all $s \in S(K)$. If $\mu \in P(K)$, the resultant $r(\mu)$ is that unique point in K with $\mu(a) = a(r(\mu))$ all $a \in A(K)$. If $\mu \prec \nu$, then $r(\mu) = r(\nu)$, and given any $\mu \in P(K)$, there is a maximal ν with $\mu \prec \nu$. If ν is maximal, then supp $\nu \subseteq Z$.

We identify $P(Z)$ with the $\mu \in P(K)$ for which supp $\mu \subseteq Z$, hence the notions of dilation and resultant are defined on $P(Z)$. We define a function $b \in C(Z)$ to be *affine on* Z, if $b(z) = \lambda(f)$ whenever $\lambda \in P(Z)$ is a maximal measure with $r(\lambda) = z$ (recall that maximal measures lie in $P(Z)$). We let $A(Z)$ be the closed subspace of affine functions in $C(Z)$.

The following is an improvement of a result of Cartier (see [21, Prop. 13.1]) in which the maximality conditions were not included. The upper, lower envelope technique was introduced by Alfsen and Skau in another situation where one wanted a Krein-Milman type theorem without the usual compactness hypotheses (see [4]).

THEOREM 2.1. *Suppose that C is a compact subset of K. Define subsets S and T of $P(K) \times P(K)$ as follows:*

$$S = \{(\mu, \nu) \colon \mu \prec \nu, \text{ supp } \mu \subseteq C, \nu \text{ maximal}\},$$

$$T = \{(\delta_p, \lambda) \colon p = r(\lambda), \ p \in C, \ \lambda \text{ maximal}\}.$$

Then S is contained in the weak closed convex hull of T.*

PROOF. Suppose that the conclusion is false. From the Hahn-Banach Theorem, there exists a weak* continuous linear functional L on $M(K) \times M(K)$, a pair $(\mu, \nu) \in S$, and a constant α, with

$$L(\delta_p, \lambda) \leqq \alpha \text{ all } (\delta_p, \lambda) \in T,$$

$$L(\mu, \nu) > \alpha.$$

There exist continuous functions $f, g \in C(K)$ with $L(\alpha, \beta) = \alpha(f) - \beta(g)$ for all $(\alpha, \beta) \in M(K) \times M(K)$. Thus

$$f(p) - \lambda(g) \leqq \alpha \text{ all } (\delta_p, \lambda) \in T,$$

$$\mu(f) - \nu(g) > \alpha.$$

The family

$$\mathscr{U}_g = \{u \in -S(K) \colon u \geqq g\}$$

is directed downwards, and the upper envelope for g is defined by

$$\bar{g} = \inf \mathscr{U}_g = \lim \{u \colon u \in \mathscr{U}_g\}.$$

Since λ and ν are maximal,

$$f(p) - \lambda(\bar{g}) \leqq \alpha \text{ all } (\delta_p, \lambda) \in T,$$

$$\mu(f) - \nu(\bar{g}) > \alpha.$$

We have

$$\nu(\bar{g}) = \lim \{\nu(u) \colon u \in \mathscr{U}_g\},$$

hence we may select $u \in -S(K)$ with $u \geqq g$ and

$$f(p) - \lambda(u) \leqq \alpha \text{ all } (\delta_p, \lambda) \in T, \tag{2.1}$$

$$\mu(f) - \nu(u) > \alpha. \tag{2.2}$$

Since u is concave and continuous, if we let $\underline{u}$ be the lower envelope, of u, i.e., the supermum of v in $S(K)$ with $v \leqq u$, we have that

$$\underline{u}(p) = \inf\{\lambda(u) : (\delta_p, \lambda) \in T\}$$

(see [21, Prop. 3.1].) From (2.1),

$$f(p) - \alpha \leqq \underline{u}(p) \text{ all } p \in C,$$

hence since $\underline{u}$ is convex, $\mu \prec \nu$, and ν is maximal,

$$\mu(f) - \alpha \leqq \mu(\underline{u}) \leqq \nu(\underline{u}) = \nu(u).$$

This contradicts (2.2).

LEMMA 2.2. *Suppose that $b \in A(Z)$, and that $\mu, \nu \in P(Z)$ are such that $\mu \prec \nu$. Then $\mu(b) = \nu(b)$.*

PROOF. Since in the general case we may choose θ maximal with $\mu \prec \nu \prec \theta$, it suffiices to assume that ν is maximal. Let $C = Z$ in the statement of Theorem 2.1. We may regard S and T as subsets of $P(Z) \times P(Z)$ with the weak* topology induced by $C(Z) \times C(Z)$. Define a function F on $P(Z) \times P(Z)$ by $F(\mu, \nu) = \mu(b) - \nu(b)$. Then F is weak* continuous and affine. Since b is affine on Z, F is zero on T, and thus on the convex hull of T. From Theorem 2.1, it follows that F is zero on S.

THEOREM 2.3. *A function $f \in C(Z)$ will lie in $A(K)$ if and only if*

(1) *$f \in A(Z)$, i.e., λ maximal and $r(\lambda) = z \in Z$ imply $\lambda(f) = f(z)$,*

(2) *if ν_1 and ν_2 are maximal probability measures with the same resultant, then $\nu_1(f) = \nu_2(f)$.*

PROOF. The necessity of these conditions is obvious. Conversely suppose that f satisfies (1) and (2). It suffices to prove that if $\theta \in M(Z)$ annihilates $A(K)$, then $\theta(f) = 0$. Let $\theta = \theta^+ - \theta^-$ be the Hahn decomposition for θ. Then since $\theta(1) = 0$, $\theta^+(1) = \theta^-(1)$ and we have that $\theta/\theta^+(1) = \mu_1 - \mu_2$ where $\mu_1, \mu_2 \in P(Z)$. Since $\mu_1 \mid A(K) = \mu_2 \mid A(K)$, $r(\mu_1) = r(\mu_2)$. Let ν_1 and ν_2 be maximal with $\mu_1 \prec \nu_1$, $\mu_2 \prec \nu_2$. From Lemma 2.2 and (2)

$$\mu_1(f) - \mu_2(f) = \nu_1(f) - \nu_2(f) = 0$$

hence $\theta(f) = 0$, and the theorem follows.

REMARK. The problem of determining which functions on $E(K)$ extend to elements of $A(K)$ was solved by Alfsen in [2]. Boboc and Bucur [6] recently proved a result that characterizes which functions in $C(Z)$ extend. The above theorem differs from [6] in that one only need use maximal measures in (1). Specifically, the Boboc and Bucur Theorem invokes the (formally) more restrictive class $A_1(Z)$ of all $f \in C(Z)$ such that $\lambda \in P(Z)$ and $r(\lambda) = z \in Z$ imply $\lambda(f) = f(z)$. Their result is closely related to the theorem of [2], since as Alfsen has pointed out to me, the functions considered in [2] are just those uniformly continuous functions on $E(K)$ whose continuous extension belongs to $A_1(Z)$. The above refinement will play an important role in the proofs of Corollary 3.3 and Theorem 5.7 and 6.3.

3. Lindenstrauss spaces

Let V be a real Banach space, and K the closed unit ball of V^* with the weak* topology. We define an affine homeomorphism σ of K by $\sigma(p) = -p$. It is clear that $E(K)$ and thus Z are invariant under σ. We let $C_\sigma(K)$ be the closed subspace of *odd* functions in $C(K)$, i.e., those $f \in C(K)$ for which $f(\sigma p) = -f(p)$, and we let $A_\sigma(K)$, $C_\sigma(Z)$, and $A_\sigma(Z)$ be the corresponding subsets of $A(K)$, $C(Z)$, and $A(Z)$, respectively. We define a map F of V into $C(K)$ by $F(v)(p) = p(v)$.

LEMMA 3.1. *F is an isometry of V onto $A_\sigma(K)$.*

PROOF. F is trivially an isometry. That it is onto is a consequence of [15, Lemma 4.3].

We shall identify V and $A_\sigma(K)$.

We define order and norm preserving transformations σ on $C(K)$ and $C(Z)$ by $\sigma(f)(p) = f(\sigma p)$. This induces corresponding transformations σ of $M(K)$, $M(Z)$, and $P(K)$ by $(\sigma\mu)(f) = \mu(\sigma f)$. We define for f and μ, odd $f = \frac{1}{2}(f - \sigma f)$, odd $\mu = \frac{1}{2}(\mu - \sigma\mu)$.

We recall that a Kakutani *L-space* is a vector lattice W with a norm satisfying

$$\| p + q \| = \| p \| + \| q \|, \quad p, q \in W^+ \tag{3.1}$$

$$\| p^+ \| + \| p^- \| = \| p \|. \tag{3.2}$$

A Banach space V is a *Lindenstrauss space* if V^* is isometric to an L-space W. We will generally regard V^* as itself an L-space. For the simplest examples, suppose that X is compact Hausdorff. Then $M(X)$ is an L-space, $C(X)$ a Lindenstrauss space.

The equivalence (1)⇔(3) of the following Theorem is due to Lazar [16]. We feel that the interpolation of (2) in the proof will help to clarify the geometric significance of the result.

THEOREM 3.2. *Let V be a real Banach space, K the closed unit ball of V^*. The following are equivalent*:

(1) *V is a Lindenstrauss space.*

(2) *If μ_1 and μ_2 are discrete probability measures on K with $r(\mu_1) = r(\mu_2)$, then there are discrete dilations ν_i of μ_i, $i = 1, 2$, for which odd $\nu_1 =$ odd ν_2.*

(3) *If* ν_1 *and* ν_2 *are maximal probability measures on* K *with* $r(\nu_1)=r(\nu_2)$, *then* *odd* $\nu_1 =$ *odd* ν_2.

PROOF. (1) $\Rightarrow$ (2). Let Q be the positive elements of norm 1 in K. Then K is the convex hull of Q and $-Q$. To see this, suppose $p \in K$, and let q be an arbitrary element of Q with $\| q \| = 1$. Then the following is a convex decomposition of p:

$$p = \| p^+ \| \frac{p^+}{\| p^+ \|} + \| p^- \| \frac{(-p^-)}{\| p^- \|} + \tfrac{1}{2}(1 - \| p \|)q + \tfrac{1}{2}(1 - \| p \|)\ (-q).$$

If p^+ or $p^- = 0$, we simply delete that term in the sum.

Suppose that

$$\mu_1 = \sum_{i=1}^{m} a_i \delta_{p_i},\ \mu_2 = \sum_{j=1}^{n} b_j \delta_{q_j}$$

where $p_i,\ q_j \in K$; $0 < a_i,\ b_j$; and $\sum a_i = \sum b_j = 1$.
Letting $p = r(\mu_1) = r(\mu_2)$, we have

$$p = \sum_{i=1}^{m} a_i p_i = \sum_{j=1}^{n} b_j q_j. \tag{3.3}$$

Choose scalars α_i, β_j with $0 \leqq \alpha_i, \beta_j \leqq 1$ and p_i', p_i'', q_j', q_j'' in Q with

$$p_i = \alpha_i p_i' + (1 - \alpha_i)(-p_i''),$$
$$q_j = \beta_j q_j' + (1 - \beta_j)(-q_j'').$$

From (3.3) we have

$$\sum_{i=1}^{m} a_i\alpha_i p_i' + \sum_{j=1}^{n} b_j(1-\beta_j)q_j'' = \sum_{i=1}^{m} a_i(1-\alpha_i)p_j'' + \sum_{j=1}^{n} b_j\beta_j q_j'.$$

Since the positive cone of V^* is lattice ordered, it satisfies the Riesz decomposition property (see [21, Lemma 9.1]). It follows that there exist $r_{i,j} \geqq 0$ with

$$a_i\alpha_i p_i' = \sum_{k=1}^{m+n} r_{i,k} \qquad i = 1, \cdots, m,$$

$$b_j(1-\beta_j)q_j'' = \sum_{k=1}^{m+n} r_{m+j,k} \qquad j = 1, \cdots, n,$$

$$a_i(1-\alpha_i)p_i'' = \sum_{k=1}^{m+n} r_{k,i} \qquad i = 1, \cdots, m,$$

$$b_j\beta_j q_j' = \sum_{k=1}^{m+n} r_{k,m+j} \qquad j = 1, \cdots, n.$$

We may select $s_{i,j} \in Q$ and scalars $c_{i,j} \geqq 0$ with $r_{i,j} = c_{i,j} s_{i,j}$. Then from additivity of the norm,

$$\alpha_i = \sum_k c_{i,k}/a_i$$

$$1 - \beta_j = \sum_k c_{m+j,k}/b_j$$

$$1 - \alpha_i = \sum_k c_{k,i}/a_i$$

$$\beta_j = \sum_k c_{k,m+j}/b_j$$

In particular

$$p_i = \sum_k \frac{c_{i,k}}{a_i} s_{i,k} + \frac{c_{k,i}}{a_i}(-s_{k,i})$$

$$q_j = \sum_k \frac{c_{m+j,k}}{b_j}(-s_{m+j,k}) + \frac{c_{k,m+j}}{b_j} s_{k,m+j}$$

provide convex decompositions of p_i and q_j.

We let

$$v_{1i} = \sum_k \frac{c_{i,k}}{a_i}\,\delta(s_{i,k}) + \frac{c_{k,i}}{a_i}\,\delta(-s_{k,i})$$

$$v_{2j} = \sum_k \frac{c_{m+j,k}}{b_j}\,\delta(-s_{m+j,k}) + \frac{c_{k,m+j}}{b_j}\,\delta(s_{k,m+j})$$

and $v_1 = \Sigma_i a_i v_{1i}$, $v_2 = \Sigma_j b_j v_{2j}$. Then since $r(v_{1i}) = p_i$ and $r(v_{2j}) = q_j$, we conclude

$$\mu_1 = \sum_i a_i \delta(p_i) \prec v_1,$$

$$\mu_2 = \sum_j b_j \delta(q_j) \prec v_2.$$

On the other hand

$$v_1 = \theta_1 + \lambda$$

$$v_2 = \theta_2 + \lambda$$

where

$$\theta_1 = \sum_{i,j=1}^{m} c_{i,j}[\delta(s_{i,j}) + \delta(-s_{i,j})]$$

$$\theta_2 = \sum_{i,j=m+1}^{m+n} c_{i,j}[\delta(s_{i,j}) + \delta(-s_{i,j})]$$

$$\lambda = \sum_{i=1}^{n} \sum_{j=m+1}^{m+n} [c_{i,j}\delta(s_{i,j}) + c_{j,i}\delta(-s_{j,i})].$$

We have odd $\theta_i = 0$, hence

$$\text{odd } \nu_1 = \text{odd } \lambda = \text{odd } \nu_2.$$

(2) $\Rightarrow$ (3). Let $\mu_{1\alpha}$ and $\mu_{2\alpha}$ be nets of discrete measures converging weak* to ν_1 and ν_2, respectively. For each α choose $\nu_{1\alpha} \succ \mu_{1\alpha}$, $\nu_{2\alpha} \succ \mu_{2\alpha}$, with odd $\nu_{1\alpha}$ = odd $\nu_{2\alpha}$. We claim that $\nu_{1\alpha}$ converges to ν_1. Let ν_1' be a limit of a convergent subnet $\nu_{1\beta}$. If s is convex and continuous, $\nu_{1\beta}(s) \geqq \mu_{1\beta}(s)$. It follows that $\nu_1'(s) \geqq \nu_1(s)$, i.e., $\nu_1' \succ \nu_1$. Since ν_1 is maximal, $\nu_1 = \nu_1'$, and from the compactness of $P_1(K)$, the convergence assertion follows. Similarly $\nu_{2\alpha}$ converges weak* to ν_2. Since for any measure ν, odd $\nu(f) = \nu(\text{odd } f)$, odd is weak* continuous. It follows that odd ν_1 = odd ν_2.

(3) $\Rightarrow$ (1). See [16].

COROLLARY 3.3. *If V is a Lindenstrauss space, then $V = A_\sigma(Z)$.*

PROOF. From Lemma 3.1, we must show that $A_\sigma(Z) \subseteq A_\sigma(K)$. Choose $f \in A_\sigma(Z)$. If ν_1 and ν_2 are maximal probability measures with $r(\nu_1) = r(\nu_2)$, we have from (3) of Theorem 3.2 that odd ν_1 = odd ν_2. Since f is odd,

$$\nu_1(f) = \text{odd } \nu_1(f) = \text{odd } \nu_2(f) = \nu_2(f).$$

From Theorem 2.3, $f \in A_\sigma(K)$.

4. Bifaces and the structure topology

Let V be a Lindenstrauss space, K the closed unit ball of V^*. Lazar and Lindenstrauss showed that theorems about the face of a simplex will generalize to K if one considers the symmetric convex set generated by a face. A more transparent theory results if one uses the equivalent notion of a "biface" of K. The latter is defined by means of a weak ordering that exists in any normed linear space.

Suppose that W is a normed linear space. We say that $p, q \in W$ are *without cancellation*, and write $p \mid q$, if $\| p + q \| = \| p \| + \| q \|$. We say that $r \in W$ *dominates* p, and write $p \prec r$, if $r = p + q$ where $p \mid q$. If $p \prec q \prec r$, then $p \prec r$. If $p \prec q$ and $q \prec p$,

$$\begin{aligned} \| q \| &= \| p \| + \| q - p \| \\ &= \| q \| + \| p - q \| + \| q - p \| \end{aligned}$$

hence $p = q$. If α is *any* scalar, then $p \prec q$ implies $\alpha p \prec \alpha q$. If $p \prec q$ and $q \mid r$, then

$$\begin{aligned}\|p+r\|+\|q-p\| &\geqq \|q+r\| \\ &= \|q\|+\|r\| \\ &= \|p\|+\|q-p\|+\|r\|,\end{aligned}$$

hence, $p \mid r$. One does not have that $p \prec q$ implies $p + r \prec q + r$.

If W is an L-space, then from (3.1), $p \mid q$ for any $p, q \in W^+$, i.e., $0 \leqq p \leqq r$ implies $p \prec r$. From (3.2), if p is any element of W, we have $p^+ \mid -p^-$, hence $p^+ \prec p$ and $-p^- \prec p$. If $p, q \geqq 0$, then since $(p-q)^+ = p - p \wedge q$ and $(p-q)^- = q - p \wedge q$,

$$\begin{aligned}\|p-q\| &= \|p - p\wedge q\| + \|q - p \wedge q\| \\ &= \|p\|+\|q\|-2\|p\wedge q\|,\end{aligned}$$

and it follows that $p \mid (-q)$ if and only if $p \wedge q = 0$. In general,

LEMMA 4.1. *Suppose that W is an L-space, and $p, q \in W$. Then the following are equivalent*:

(1) $p \mid q$

(2) $p^+ \wedge q^- = p^- \wedge q^+ = 0$

(3) $(p+q)^+ = p^+ + q^+$ *and* $(p+q)^- = p^- + q^-$.

PROOF. (1) $\Rightarrow$ (2). If $p \mid q$, then since $p^+ \prec p$ and $-q^- \prec q$, $p^+ \mid -q^-$, and from above, $p^+ \wedge q^- = 0$. A similar argument shows that $p^- \wedge q^+ = 0$.

(2) $\Rightarrow$ (3). In general we have that $a \wedge b = 0$ implies that

$$(a-b)^+ = a - a \wedge b = a$$

$$(a-b)^- = b - a \wedge b = b.$$

From (2) we have

$$(p^+ + q^+) \wedge (p^- + q^-) = 0.$$

Since $p + q$ is the difference of these terms,

$$(p+q)^+ = p^+ + q^+,$$

$$(p+q)^- = p^- + q^-.$$

(3) $\Rightarrow$ (1). We have

$$\begin{aligned}\|p+q\| &= \|(p+q)^+\| + \|(p+q)^-\| \\ &= \|p^+ + q^+\| + \|p^- + q^-\| \\ &= \|p^+\| + \|q^+\| + \|p^-\| + \|q^-\| \\ &= \|p\| + \|q\|.\end{aligned}$$

COROLLARY 4.2. *Suppose that W is an L-space, and p, $q \in W$. Then $p \prec q$ if and only if $p^+ \leqq q^+$ and $p^- \leqq q^-$.*

PROOF. We have $p \prec q$ if and only if $p \mid q - p$, i.e., from (3), if and only if

$$q^+ = p^+ + (q-p)^+$$
$$q^- = p^- + (q-p)^-.$$

If the equations hold, $q^+ \geqq p^+$ and $q^- \geqq p^-$. Conversely if $q^+ \geqq p^+$ and $q^- \geqq p^-$, then

$$(q-p)^+ = ((q^+ - p^+) - (q^- - p^-))^+ = q^+ - p^+,$$

and similarly, $(q-p)^- = q^- - p^-$.

LEMMA 4.3. *Suppose that W is an L-space, and p, $q \in W$. Then there exist p_1 and q_1 with $p_1 \prec p$, $q_1 \prec q$, $p_1 \mid q_1$ and*

$$p + q = p_1 + q_1.$$

PROOF. We have that

$$\begin{aligned} p+q &= p^+ - q^- + q^+ - p^- \\ &= p_1 + q_1, \end{aligned}$$

where

$$p_1 = (p^+ - q^-)^+ - (q^+ - p^-)^-$$
$$q_1 = (q^+ - p^-)^+ - (p^+ - q^-)^-.$$

We have

$$p_1^+ \leqq (p^+ - q^-)^+ \leqq p^+$$
$$p_1^- \leqq (q^+ - p^-)^- \leqq p^-,$$

hence from Corollary 4.2, $p_1 \prec p$. A similar calculation shows that $q_1 \prec q$. From (2) of Lemma 4.1, it is immediate that $p_1 \mid q_1$.

LEMMA 4.4 (RIESZ DECOMPOSITION PROPERTY). *Suppose that W is an L-space, and that p, q_1, $q_2 \in W$ are such that*

$$p \prec q_1 + q_2, \; q_1 \mid q_2.$$

Then there exist p_1 and p_2 with

$$p = p_1 + p_2, \; p_i \prec q_i.$$

PROOF. From Corollary 4.2 and Lemma 4.1 (3),

$$p^+ \leqq (q_1 + q_2)^+ = q_1^+ + q_2^+,$$
$$p^- \leqq (q_1 + q_2)^- = q_1^- + q_2^-.$$

Since W is lattice ordered, it satisfies the usual Riesz decomposition property (see [21, Lemma 9.1]), hence there exist r_1, r_2, s_1, s_2 with

$$p^+ = r_1 + r_2, \quad 0 \leqq r_i \leqq q_i^+$$

$$p^- = s_1 + s_2, \quad 0 \leqq s_i \leqq q_i^-.$$

Letting $p_i = r_i - s_i$, we have $p = p_1 + p_2$. Since

$$r_i \wedge s_i \leqq q_i^+ \wedge q_i^- = 0,$$

we have

$$p_i^+ = r_i \leqq q_i^+$$

$$p_i^- = s_i \leqq q_i^-$$

hence $p_i \prec q_i$.

Suppose that W is a normed space, and K is its closed unit ball. A non-empty subset H of K is a *biface* in K if

$B_1 \cdot H$ is convex and symmetric.

$B_2 \cdot$ If $x \neq 0$ is in H, then so is $x/\| x \|$.

$B_3 \cdot$ If $q \in H$ and $p \prec q$, then $p \in H$.

It is readily verified from B_1 and B_2 that H must be the unit ball of the normed space lin H, the linear span of H. Thus we may consider bifaces in H. Any biface in H is a biface in K. If $p \in E(K)$, then the reader may verify that the *extremal diameter* $H_p = [-p, p]$ is a biface in K. If Q is a face in K, the convex hull of Q and $-Q$ need not be a biface (consider the hexagon), although this is the case if W is an L-space. A. Lazar has pointed out to the author that every non-zero biface of an L-space arises in this manner. We will not need these simple results.

LEMMA 4.5. *Let K be the unit ball of a normed linear space W. If $H \neq \{0\}$ is a biface in K, then $E(H) \subset E(K)$.*

PROOF. Suppose that $p \in E(H)$, and

$$p = \alpha q + (1 - \alpha)r, q, r \in K, \qquad 0 < \alpha < 1.$$

Since $\| p \| = 1 \geqq \| q \|, \| r \|$, we have $\alpha q \,|\, (1 - \alpha)r$, hence from B_3 and B_2, $\alpha q \in H$, and $q \in H$. Similarly $r \in H$, and we conclude $p = q = r$.

The intersection of bifaces is again a biface. The convex hull of bifaces is in general not a biface (consider the hull of two extremal diameters in a hexagon).

PROPOSITION 4.6. *Let K be the unit ball of an L-space W. If H_1 and H_2 are bifaces in K, so is the convex hull $H = c(H_1, H_2)$.*

PROOF. H trivially satisfies B_1. If $p \in H$, we have $p = \alpha_1 p_1 + \alpha_2 p_2, \alpha_1 + \alpha_2 = 1$, $\alpha_i \geqq 0$, and $p_i \in H_i$. It follows that $\alpha_i p_i \in H_i$. From Lemma 4.3 we may choose $q_i \prec \alpha_i p_i$ with $q_1 | q_2$ and $p = q_1 + q_2$. Thus $q_i / \| q_i \| \in H_i$,

$$\frac{p}{\| p \|} = \frac{\| q_1 \|}{\| p \|} \frac{q_1}{\| q_1 \|} + \frac{\| q_2 \|}{\| p \|} \frac{q_2}{\| q_2 \|} \in H,$$

and we have B_2. Finally, if $0 \neq r \prec p$, then from Lemma 4.4, there exist $r_i \prec q_i$ with $r = r_1 + r_2$. Since $r_1 | r_2$, $r / \| r \| \in H$ (see the above calculation), and $r \in H$.

In the remainder of this section, V will be a Lindenstrauss space and K, the unit ball of V^*. If $H_1 \neq \{0\}$ and $H_2 \neq \{0\}$ are weak* closed bifaces in K, the same is true for $H = c(H_1, H_2)$, and since H_i is a biface in H, we have

$$E(H) = E(H_1) \cup E(H_2).$$

If H_α are weak*-closed bifaces and $\{0\} \neq H = \cap H_\alpha$, then

$$E(H) = \cap E(H_\alpha).$$

We may thus define a subset F of $E(K)$ to be *structurally closed* if $F = E(H)$, $H \neq \{0\}$ a weak* closed biface, or if $F = \varnothing$. Points in $E(K)$ are never closed in the *structure topology*, since an exteme point can never be separated from its negative. For convenience of terminology, we define $E_\sigma(K)$ to be the sets $\{p, -p\}$ with $p \in E(K)$, and we define the structure topology on $E_\sigma(K)$ to be the quotient topology. Since the sets H_p, $p \in E(K)$ are bifaces, points in $E_\sigma(K)$ are closed.

If p is an arbitrary point in K, we let H_p be the smallest weak* closed biface containing p. We conclude this section with some simple properties of the structure topology.

PROPOSITION 4.7. *If a net p_α in $E(K)$ converges to a point $p \in K$, $p \neq 0$, then p_α converges in the structure topology to each $q \in E(H_p)$.*

PROOF. Suppose that $q \in E(H_p)$ and that p_α does not converge to q. Then there is a subnet p_β and a structure closed set F with $p_\beta \in F$, $q \notin F$. Letting $F = E(H)$, H a weak* closed biface, $p_\beta \in H$ imply that $p \in H$, hence $q \in E(H_p) \subseteq E(H)$, a contradiction.

PROPOSITION 4.8 (see [13, p. 43]). *If f is a continuous, convex function on K with $f(0) = 0$, then the restriction $f | E(K)$ vanishes at ∞, i.e., for each $c > 0$, the set*

$$D = \{p \in E(K) : f(p) \geqq c\}$$

is structurally compact.

PROOF. Let p_α be a net in D. Then let p_β be a subnet converging weak* to $q \in K$. Since f is continuous, $f(q) \geqq c$, and in particular $q \neq 0$. Since f is convex, $f \mid H_q$ will assume a maximum value at some point $p \in E(H_q)$ (see [5, p. 7]), i.e., $f(p) \geqq c$, and $p \in D$. From Proposition 4.7, p_β converges structurally to p.

PROPOSITION 4.9. *If $v \in V$, then the set*

$$\{p \in E(K) : |v(p)| = \|v\|\}$$

is structurally closed.

PROOF. It suffices to show that

$$H = \{p \in K : |v(p)| = \|v\| \; \|p\|\}$$

is a biface. It is clear that H satisfies B_1 and B_2. If $q \prec p \in H$, then letting $p = q + r$ the inequalities

$$\begin{aligned} |v(q)| &\leqq \|v\| \; \|q\| \\ |v(r)| &\leqq \|v\| \; \|r\| \\ \|v\|(\|q\| + \|r\|) &= \|v\| \; \|q + r\| \\ &= |v(p)| \\ &\leqq |v(q)| + |v(r)| \end{aligned}$$

imply that $|v(q)| = \|v\| \; \|q\|$.

5. Continuous multipliers and symmetrically dilated sets

In this section, V will be a Lindenstrauss space, K the closed unit ball of V^*.

If θ is an odd signed measure on K, i.e., $\sigma\theta = -\theta$, a simple argument with the Hahn decomposition for θ will show that $\sigma\theta^+ = \theta^-$, hence $\theta = \text{odd } (2\theta^+)$. If $p \in K$, we define

$$\omega_p = 2(\text{odd } \mu)^+,$$

where μ is any maximal probability measure representing p. This is well-defined by Lazar's criterion (Theorem 3.2). We note that for $v \in V$, $\omega_p(v) = v(p)$, and

$$0 \leqq \omega_p = (\mu - \sigma\mu)^+ = \mu - \mu \wedge \sigma\mu \leqq \mu. \tag{5.1}$$

LEMMA 5.1. *If $0 \leqq c \leqq 1$, then $\omega_{cp} = c\omega_p$.*

PROOF. Suppose that μ is a maximal probability measure with resultant p. Let q be an arbitrary element in $E(K)$. Then

$$\nu = c\mu + \tfrac{1}{2}(1-c)(\delta_q + \delta_{-q})$$

is a maximal probability measure with $r(\nu) = cp$. We have $\sigma(\delta_q) = \delta_{-q}$, hence odd $\nu = c$ odd μ, and

$$\omega_{cp} = 2(\operatorname{odd}\nu)^+ = 2c(\operatorname{odd}\mu)^+ = c\omega_p.$$

We let $M_1^+(K)$ be the measures μ with $\mu \geqq 0$, $\|\mu\| \leqq 1$. If $\mu \in M_1^+(K)$ we define the *resultant*, $r(\mu)$ to be the unique $p \in K$ with $\mu(v) = v(p)$ for all $v \in V$. In particular $r(\omega_p) = p$ for each $p \in K$. We say that μ is *maximal* if $\mu \neq 0$ and $\mu/\|\mu\|$ is a maximal probability measure, or $\mu = 0$. If μ is maximal, and $0 \leqq \nu \leqq \mu$, then ν is maximal. This is a consequence of the fact that the maximal probability measures form a face in $P(K)$, (this is clear from [21, Prop. 9.3]), and if $\mu \neq 0$,

$$\frac{\mu}{\|\mu\|} = \frac{\|\nu\|}{\|\mu\|}\frac{\nu}{\|\nu\|} + \frac{\|\mu-\nu\|}{\|\mu\|}\frac{\mu-\nu}{\|\mu-\nu\|}.$$

From (5.1) ω_p must be maximal for each $p \in K$.

LEMMA 5.2. *Let $\theta \in M_1^+(K)$ be maximal and $p = r(\theta)$. The following are equivalent*:

(1) $\theta = \omega_p$

(2) $\|\theta\| = \|p\|$

(3) $\theta \wedge \sigma\theta = 0$.

PROOF. Let q be an arbitrary element of $E(K)$. Then

$$\mu = \|\theta\|\frac{\theta}{\|\theta\|} + \tfrac{1}{2}(1 - \|\theta\|)(\delta_q + \delta_{-q})$$

is a maximal probability measure with $r(\mu) = p$ (delete the first term if $\theta = 0$). Thus

$$\omega_p = 2(\operatorname{odd}\mu)^+ = (\theta - \sigma\theta)^+ = \theta - \theta \wedge \sigma\theta. \tag{5.2}$$

The equivalence of (1) and (3) is thus immediate. Since p is just the restriction of ω_p to V, $\|p\| \leqq \|\omega_p\|$. On the other hand from Lemma 5.1, if $p \neq 0$,

$$\|\omega_p\| = \|p\| \, \|\omega_{p/\|p\|}\| \leqq \|p\|.$$

If $p = 0$, then choosing q as above,

$$\omega_p = 2(\operatorname{odd}\tfrac{1}{2}(\delta_q + \delta_{-q}))^+ = 0,$$

and in general, $\|\omega_p\| = \|p\|$. Since

$$\| \theta - \theta \wedge \sigma\theta \| = \| \theta \| - \| \theta \wedge \sigma\theta \|,$$

the equivalence of (2) and (3) follows when one takes the norms in (5.2).

COROLLARY 5.3. *If p, q, $p + q \in K$, then $p \mid q$ if and only if $\omega_{p+q} = \omega_p + \omega_q$.*

PROOF. $\omega_p + \omega_q$ is a maximal measure, and $r(\omega_p + \omega_q) = p + q$. Thus $\omega_p + \omega_q = \omega_{p+q}$ if and only if

$$\| \omega_p + \omega_q \| = \| p + q \|,$$

or equivalently, $\| p \| + \| q \| = \| p + q \|$, i.e., $p \mid q$.

COROLLARY 5.4. *If p, $q \in K$, then $p \prec q$ if and only if $\omega_p \leqq \omega_q$.*

PROOF. *If $p \prec q$*, then $q - p \in K$ and $p \mid q - p$, hence

$$\omega_q = \omega_p + \omega_{q-p} \geqq \omega_p.$$

If $\omega_q \geqq \omega_p$, then $\omega_q - \omega_p$ is a maximal measure, $r(\omega_q - \omega_p) = q - p$, and $\| \omega_q - \omega_p \| = \| q \| - \| p \| \leqq \| q - p \|$. It follows that $\omega_q - \omega_p = \omega_{q-p}$, hence

$$\omega_q = \omega_p + \omega_{q-p}$$

and $p \mid q - p$.

I am indebted to F. Perdrizet for a correction in the proof of the following result.

LEMMA 5.5. *If J is a compact convex subset of K, and μ is a maximal measure on K with supp $\mu \subseteq J$, then μ is maximal when regarded as a measure on J.*

PROOF. We may assume that μ is a probability measure on J. Suppose that ν is another probability measure on J with $\mu(s) \leqq \nu(s)$ for all $s \in S(J)$. Then $\mu(t) \leqq \nu(t)$ for all $t \in S(K)$, and regarding μ and ν as measures on K, $\mu \prec \nu$. Since μ is maximal on K, $\mu = \nu$.

LEMMA 5.6. *If $0 \neq p \in K$, then supp $\omega_p \subseteq \overline{E(H_p)}$.*

PROOF. Since $\omega_p = \| p \| \omega_{p/\|p\|}$ for $p \neq 0$, we may assume that $\| \omega_p \| = \| p \| = 1$. Let $B_1, \cdots, B_N$ be disjoint Borel sets with $K = \cup B_n$ and $c_n = \omega_p(B_n) \neq 0$. Let $\mu_n = \omega_p \mid B_n$ be the restriction of ω_p to B_n, θ_n the probability measure μ_n / c_n, and

$p_n = r(\theta_n)$. Let μ be the "discrete approximation" $\Sigma c_n \delta(p_n)$. We claim that supp $\mu \subseteq H_p$.

We have that $p = \Sigma c_n p_n$. Since $\| p_n \| \leqq 1$ for each n and

$$1 = \| p \| \leqq \Sigma c_n \| p_n \|,$$

we have that $\| p_n \| = 1$ for each n. On the other hand,

$$1 = \| p \| \leqq \| c_n p_n \| + \| \Sigma_{k \neq p} c_k p_k \| \leqq \Sigma c_k \| p_k \| = 1,$$

hence $c_n p_n \prec p$. From B_3 and B_2 (see §4), $c_n p_n \in H_p$ and $p_n \in H_p$, hence supp $\mu \leqq H_p$.

We have that there is a net of "discrete approximations" μ_α converging weak* to ω_p (see [3, Prop. *I*. 2.3]). Since H_p is compact, supp $\omega_p \subseteq H_p$. From Lemma 5.5 ω_p is maximal as a measure on H_p, hence supp $\omega_p \subseteq \overline{E(H_p)}$ (see [21, p. 30]).

Dauns and Hofmann proved that continuous bounded functions on the structure space of a C^*-algebra act as "multipliers" for the C^*-algebra. The following is an analogous result for Lindenstrauss spaces (see [11, Th. 2.1; 3, §7] for parallel results). Let $C_s(E(K))$ be the bounded, structurally continuous real functions on $E(K)$. Note that such functions f must be even, i.e., $f(-p) = f(p)$ for all $p \in E(K)$.

THEOREM 5.7. *Suppose that V is a Lindenstrauss space, and K is the closed unit ball of V^*. If $f \in C_s(E(K))$ and $v \in V$, then there exists an element $w \in V$ with $w(p) = f(p)\, v(p)$ for all $p \in E(K)$.*

PROOF. We first extend f to a function on $Z - \{0\}$. If $q \in Z - \{0\}$, and $p_\alpha \in E(K)$ converges to q, then fixing $p_0 \in E(H_q)$, we have that p_α converges to p_0 structurally (Proposition 4.7), hence $f(p_\alpha)$ converges to $f(p_0)$. The latter value does not depend on the net p_α. Thus we define $\tilde{f}(q) = f(p_0)$. From [24, p. 100, Problem D], it follows that $\tilde{f}$ is continuous on $Z - \{0\}$. We define $\tilde{f}(0) = 0$ (this will usually introduce a discontinuity). Since f is bounded and even, the same is true for $\tilde{f}$.

If $v \in V$, $v(0) = 0$, hence $\tilde{f}v$ is a continuous odd function on Z. To show that it is the restriction of an element w of V, it suffices to show that $\mu(\tilde{f}v) = \tilde{f}(p)\, v(p)$ for each $p \in Z$ and maximal probability measure μ representing p (Corollary 3.3) Since $\tilde{f}v$ is odd,

$$\begin{aligned} \mu(\tilde{f}v) &= \text{odd } \mu(\tilde{f}v) \\ &= 2(\text{odd } \mu)^+ (\tilde{f}v) \\ &= \omega_p(\tilde{f}v). \end{aligned}$$

The set

$$Y_p = \{q \in Z : (\tilde{f}v)(q) = \tilde{f}(p)\, v(q)\}$$

is closed and contains $EH(p)$ since f is constant on the latter set (this is clear from the definition of $\tilde{f}$). From Lemma 5.6, if $p \neq 0$,

$$\operatorname{supp} \omega_p \subseteq E(H_p)^- \subseteq Y_p.$$

It follows that for any p,

$$\mu(\tilde{f}v) = \tilde{f}(p)\,\omega_p(v) = \tilde{f}(p)\,v(p).$$

We say that a subset D of K is *symmetrically dilated* if for each $p \in D$, $E(H_p) \subseteq D$. The following result will play an important role in Sections 6 and 7.

THEOREM 5.8 (see [11, TH. 3.3]): *Suppose that V is a Lindenstrauss space and that D is a compact symmetrically dilated subset of K. Then the closed convex hull $H = c(D)$ is a biface in K, and $D \cap E(K)$ is structurally closed.*

PROOF. The second assertion will follow from the first since $H \cap E(K) = D \cap E(K)$ (see [21, p. 9]). For the first it suffices to show that if $p \in H$, then $\operatorname{supp} \omega_p \subseteq H$, since then for $p \neq 0$,

$$\operatorname{supp} \omega_{p/\|p\|} = \operatorname{supp} \frac{1}{\|p\|}\,\omega_p \subseteq H,$$

hence $p/\|p\| = r(\omega_{p/\|p\|}) \in H$. If $q \prec p$, $\omega_q \leqq \omega_p$ (Corollary 5.4), hence $\operatorname{supp} \omega_q \subseteq H$ and $q \in H$.

Let $p = r(\mu)$ where $\mu \in P(D)$ (e.g., let μ be maximal on H), and v a maximal measure on K with $\mu \prec v$. From Theorem 2.1, we may find nets of probability measures μ_α, v_α converging weak* to μ and v, respectively, with

$$\mu_\alpha = \sum_{i=1}^{n_\alpha} c_i^\alpha \delta(p_i^\alpha), \qquad p_i^\alpha \in D,$$

$$v_\alpha = \sum_{i=1}^{n_\alpha} c_i^\alpha \lambda_i^\alpha,$$

where λ_i^α is maximal on K with $r(\lambda_i^\alpha) = p_i^\alpha$. We have

$$\omega_{p_i^\alpha} = 2(\operatorname{odd} \lambda_i^\alpha)^+,$$

and from Lemma 5.6,

$$\operatorname{supp} \omega_{p_i^\alpha} \subseteq \overline{E(H_{p_i^\alpha})}.$$

Since D is symmetrically dilated, $\overline{E(H_{p_i^\alpha})} \subseteq D$, and

$$\operatorname{supp}(\operatorname{odd} \lambda_i^\alpha) \subseteq D \subseteq H.$$

Thus

$$\mathrm{supp}(\mathrm{odd}\, \nu_\alpha) \subseteq H,$$

and since odd ν_α converges to odd ν,

$$\mathrm{supp}(\mathrm{odd}\, \nu) \subseteq H$$

(see [7, Ch. III. §2, Prop. 6]). Since $\omega_p = 2(\mathrm{odd}\, \nu)^+$, supp $\omega_p \subseteq H$.

COROLLARY 5.9 (see [13, P. 26]). *Let F be a subset of $E(K)$. Then the following are equivalent:*

(1) *F is structurally closed.*

(2) *F is relatively weak* closed in $E(K)$, and $\bar{F}$ is a symmetrically dilated set.*

PROOF. (1) ⇒ (2). Let $F = E(H)$, H a closed biface. Since $F = H \cap E(K)$, F is weak* relatively closed. If $p \in \bar{F}$, then $H_p \subseteq H$ and

$$E(H_p) \subseteq E(H) = F,$$

hence $\bar{F}$ is dilated.

(2) ⇒ (1). We have from Theorem 5.8 that $F = \bar{F} \cap E(H)$ is structurally closed.

COROLLARY 5.10. *If $D \subseteq E(K)$ is a weak* compact symmetric set, then it is structurally closed.*

COROLLARY 5.11 (see [22, PROP. 3.3]). *Let f be a bounded real function on $E(K)$. Then the following are equivalent:*

(1) *f is continuous in the structure topology,*

(2) *f has a weak* continuous extension $\tilde{f}$ on $Z - \{0\}$ such that $\tilde{f}(q) = f(p)$ for all $q \in Z - \{0\}$, and $p \in E(H_q)$.*

PROOF. (1) ⇒ (2). This was shown in the proof of Theorem 5.7.

(2) ⇒ (1). For any closed set $F \subseteq \mathbf{R}$,

$$D = \{q \in Z - \{0\} : \tilde{f}(q) \in F\} \cup \{0\}$$

is symmetrically dilated and weak* closed.

Letting $H = c(D)$,

$$\begin{aligned} H \cap E(K) &= D \cap E(K) \\ &= \{p \in E(K) : f(p) \in F\} \end{aligned}$$

is structurally closed.

6. *G*-spaces

A Banach space V is said to be a *G-space* if it can be mapped isometrically onto a subspace A of $C(X)$, X compact Hausdorff, of the form

$$A = \{f \in C(X): f(x_\alpha) = c_\alpha f(y_\alpha)\}$$

where $x_\alpha,\ y_\alpha \in X$ and $-1 \leqq c_\alpha \leqq 1$. The G-spaces are Lindenstrauss spaces, and may be characterized as the Banach spaces V such that given $v_1,\ v_2,\ v_3 \in V$ there is an element $v_4 \in V$ such that

$$v_4(p) = \min_{1 \leqq i \leqq 3} v_i(p) + \max_{1 \leqq i \leqq 3} v_i(p) \tag{6.1}$$

for all $p \in E(K)$, where K is the closed unit ball of V^* [19, Th. 2]. v_4 is, of course, determined by its values on $E(K)$.

(6.1) may be used to introduce various operations into V. Define for real scalars $\alpha,\ \beta,\ \gamma$, the *intermediate value* $\mathrm{Int}(\alpha,\beta,\gamma)$ to be that member of the set $\{\alpha,\beta,\gamma\}$ which lies between the other two values. Noting the identity

$$\mathrm{Int}(\alpha,\beta,\gamma) = \alpha + \beta + \gamma - [\min(\alpha,\beta,\gamma) + \max(\alpha,\beta,\gamma)] \tag{6.2}$$

we may define $\mathrm{Int}(v_1,v_2,v_3)$ for $v_1,v_2,v_3 \in V$ to be that element of V with

$$\mathrm{Int}(v_1,v_2,v_3)\,(p) = \mathrm{Int}(v_1(p),v_2(p),v_3(p)) \tag{6.3}$$

for $p \in E(K)$. We also introduce the notation

$$\alpha \curlywedge \beta = \mathrm{Int}(\alpha,\beta,0) = \begin{cases} \alpha \wedge \beta, & 0 \leqq \alpha,\ \ 0 \leqq \beta, \\ \alpha \vee \beta, & \alpha \leqq 0,\ \ \beta \leqq 0, \\ 0, & \alpha\beta \leqq 0, \end{cases}$$

and for $v_1,\ v_2 \in V$, we let

$$v_1 \curlywedge v_2 = \mathrm{Int}(v_1,v_2,0).$$

We say that an element $p \in V^*$ is a *G-character* if for all $v_1,v_2 \in V$,

$$p(v_1 \curlywedge v_2) = p(v_1) \curlywedge p(v_2). \tag{6.4}$$

LEMMA 6.1. *Let V be a Lindenstrauss space, K the closed unit ball of V^*, D a compact subset of $E(K)$ with $D \cap \sigma D = \varnothing$. If f is a real continuous function on D, then it has an isometric extension to an element of V.*

PROOF. Let Q be the closed convex hull of D. From [16, Corollary], Q is a face in K. That argument also shows that Q is a simplex, since if μ and ν are measures on $E(Q) = D$ with the same resultant, $\mu - \nu$ is an even measure with support in the asymmetric set D, hence $\mu - \nu = 0$. Since $E(Q)$ is closed, we may extend f to a continuous affine function a on Q (see [25, Satz 13]). Extend a to a function a_0 on the set $[0,1]\ Q$ by $a_0(\alpha p) = \alpha a_0(p)$. To prove a_0 is well-defined, it suffices to show that for all $p \in Q$, $\| p \| = 1$, since then $\alpha p = \beta q$, $q \in Q$ implies that $\alpha = \beta$, hence $p = q$, or $\alpha = 0$. If $p \in Q$ and $\| p \| < 1$, then drawing the chord through p and arbitrary point $q \in K$, p is a proper convex combination of q and another point, hence $q \in Q$ and $K = Q$. This contradicts the fact that $\sigma D \cap E(Q) = \varnothing$. It is clear that a_0 is affine on $[0,1]\ Q$.

The linear span lin Q is weak* closed ([17, Lemma 2.1]), hence if we let $Q^0 = \{v \in V : v \,|\, Q = 0\}$, we have $\operatorname{lin} Q \cong (V/Q^0)^*$. From [17, Lemma 2.1], the closed unit ball of lin Q is the convex hull of Q and σQ. The hypotheses of [15, Lemma 4.3] are thus satisfied and we may extend a_0 to a weak* continuous linear function b on lin Q. Since $\| b \|$ is determined by the values of b on $c(Q, -Q)$, we have $\| b \| = \| a \| = \| f \|$. Defining $g(p) = \| f \|$ for all $p \in K$, we may now apply the Lazar-Lindenstrauss-Edwards Extension Theorem [17, Th. 2.1] to obtain an isometric extension of b to an element of V.

LEMMA 6.2. *If V is a separable G-space, then the G-characters are the elements in $\mathbf{R}E(K)$.*

PROOF. From (6.3) we have that each $p \in E(K)$ is a G-character. Since we have

$$c\alpha \curlywedge c\beta = c(\alpha \curlywedge \beta)$$

for any scalar c, it follows that cp is a G-character.

Conversely suppose that p is a G-character. We may assume that $\| p \| = 1$. It suffices to show that the closed interval $[-p, p]$ is a biface in K, since then $p \in E(K)$ (Lemma 4.5). Thus it suffices to show that if $q \prec p$ or $q \prec -p$, then $q \in [-p, p]$, i.e., $p(v) = 0$ implies $q(v) = 0$. Since $-p$ is also a G-character, it suffices to assume $q \prec p$, and that $p(v) = 0$, where $\| v \| = 1$.

Since V is separable, $E(K)$ is a G_δ set, and $\omega_p(E(K)) = 1$. Since $\omega_p \wedge \sigma\omega_p = 0$ (see §5) we may find a Borel set $B \subseteq E(K)$ with $\omega_p(B) = 1$ and $B \cap \sigma B = \varnothing$. Given $\varepsilon > 0$, choose a compact subset D of B with $\omega_p(D) \geqq 1 - \varepsilon$.

$(v \,|\, D)^+$ is a continuous function on D and from Lemma 6.1 it may be extended isometrically to an element v^+ of V. Let $v^- = v^+ - v$. We next "disjoint" v^+ and v^- by defining

$$w^+ = v^+ - v^+ \curlywedge v^-,$$

$$w^- = v^- - v^+ \curlywedge v^-.$$

If $r \in E(K)$ and $v^+(r)$, $v^-(r)$ have the same sign, then since $v^+(r) \curlywedge v^-(r)$ is either $v^+(r)$ or $v^-(r)$, we have either $w^+(r) = 0$ or $w^-(r) = 0$. Since $w^+(r)$ has the same sign as $v^+(r)$ and the same is true for $w^-(r)$, $v^-(r)$, we conclude that for all $r \in E(K)$, $w^+(r)$ and $w^-(r)$ have opposite signs, i.e., $w^+ \curlywedge w^- = 0$. If $r \in D$, $v^+(r) = \max(v(r),0)$ and $v^-(r) = \max(-v(r), 0)$, hence $(v^+ \curlywedge v^-)(r) = 0$ and

$$w^+ \big| D = v^+ \big| D = (v \big| D)^+,$$
$$w^- \big| D = (v^+ - v) \big| D = (v \big| D)^-.$$

We have that

$$w^+ - w^- = v^+ - v^- = v$$

hence $p(w^+) - p(w^-) = 0$. Since p is a G-character,

$$p(w^+) \curlywedge p(w^-) = p(w^+ \curlywedge w^-) = 0.$$

Thus $p(w^+)$ and $p(w^-)$ have opposite signs, and we conclude

$$p(w^+) = p(w^-) = 0,$$

i.e.,

$$\omega_p(w^+) = \omega_p(w^-) = 0.$$

On the other hand since $q \prec p$, $\omega_q \leqq \omega_p$ (Corollary 5.4). We cannot conclude $\omega_q(w^+) = \omega_q(w^-) = 0$ since w^+ and w^- are not positive. However we do have that they are positive on D.

We have

$$\| w^+ \| \leqq \| v^+ \| = \| (v \big| D)^+ \| \leqq \| v \| = 1,$$

$$\| w^- \| = \| w^+ - v \| \leqq 2,$$

and since $\omega_p(K - D) < \varepsilon$,

$$0 \leqq \omega_q(w^+ \big| D) \leqq \omega_p(w^+ \big| D) \leqq \varepsilon,$$
$$0 \leqq \omega_q(w^- \big| D) \leqq \omega_p(w^- \big| D) \leqq 2\varepsilon,$$
$$\big| \omega_q(v \big| D) \big| \leqq 2\varepsilon,$$
$$\big| q(v) \big| = \big| \omega_q(v) \big| \leqq 2\varepsilon + \| v \| \omega_p(K - D) \leqq 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $q(v) = 0$.

THEOREM 6.3. *Suppose that V is a Lindenstrauss space, and consider the statements:*

(1) *The structure topology on* $E_\sigma(K)$ *is Hausdorff.*

(2) *The closure* Z *of* $E(K)$ *is contained in* $[0,1]E(K)$.

(3) V *is a G-space.*

One has that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *If* V *is separable, then* (3) $\Rightarrow$ (1).

PROOF. (1) $\Rightarrow$ (2). Suppose that $p \in Z$ and that $p \notin [0,1]E(K)$. Then $H_p \neq [-p/\|p\|, p/\|p\|]$ since equality would imply $p/\|p\| \in E(K)$. It is clear that $E(H_p)$ must contain linearly independent q, $r \in E(K)$. Let p_α be a net in $E(K)$ converging to p. Then from Proposition 4.7, p_α converges in the structure topology to both q and r. It follows that $E_\sigma(K)$ is not Hausdorff.

(2) $\Rightarrow$ (3). From Corollary 3.3, V is isometric to $A_\sigma(Z)$, the continuous odd affine functions on Z. If $0 \neq z \in Z$, then

$$z = \frac{1+\|z\|}{2}\frac{z}{\|z\|} + \frac{1-\|z\|}{2}\frac{(-z)}{\|z\|}$$

hence z is the resultant of the probability measure

$$\mu_z = \tfrac{1}{2}(1+\|z\|)\delta(z/\|z\|) + \tfrac{1}{2}(1-\|z\|)\delta(-z/\|z\|).$$

From (2), the latter measure is extremal. If v is any other extremal probability measure with $r(v) = z$, then odd $\mu_z =$ odd v. It follows that a function $f \in C_\sigma(Z)$ is affine if and only if

$$f(z) = \mu_z(f) = \|z\| f(z/\|z\|).$$

A may thus be identified with the $f \in C(Z)$ satisfying $f(-z) = -f(z)$ for all $z \in Z$, as well as the above relations.

(3) $\Rightarrow$ (2) (V separable). It is clear from (6.4) that the G-characters of V form a closed set. Since they contain $E(K)$, (2) follows from Lemma 6.2.

(2) $\Rightarrow$ (1) (V separable). Define a map Φ of $Z - \{0\}$ onto $E_\sigma(K)$ by

$$\Phi(p) = \{p/\|p\|, -p/\|p\|\}$$

and let R be the corresponding equivalence relation pRq if $\Phi(p) = \Phi(q)$. It suffices to show that Φ induces a homeomorphism of $Z - \{0\}/R$ onto $E_\sigma(K)$, and that R is a closed subset of $(Z - \{0\}) \times (Z - \{0\})$, since then we may apply [8, p. 112, Exercise 15] to conclude $E_\sigma(K)$ is Hausdorff. It is trivial to prove that R is closed (see [11, Lemma 3.7]). For the quotient result it suffices to prove that a set F in $E_\sigma(K)$ is closed if and only if $\Phi^{-1}(F)$ is closed in $Z - \{0\}$.

If F is closed in $E_\sigma(K)$, its inverse image F_1 in $E(K)$ is structurally closed, and there is a closed biface H in K with $F_1 = E(H)$. We have that $\Phi^{-1}(F) \subseteq [0,1]E(H)$.

Suppose that p_α is a net in $\Phi^{-1}(F)$ converging to $q \in Z - \{0\}$. Then $q \in H$ implies $q/\|q\| \in F_1$, hence $\Phi(q) = \{\pm q/\|q\|\} \in F$.

Conversely suppose that $\Phi^{-1}(F)$ is closed in $Z - \{0\}$. If $p \in \Phi^{-1}(F)$, $p \neq 0$, then $\pm p/\|p\| \in Z - \{0\}$,

$$\Phi(\pm p/\|p\|) = \Phi(p) \in F,$$

and

$$E(H_p) = \{\pm p/\|p\|\} \subseteq \Phi^{-1}(F).$$

Thus $\Phi^{-1}(F) \cup \{0\}$ is closed and symmetrically dilated. We have that $H = c(\Phi^{-1}(F) \cup \{0\})$ is a closed biface, and

$$E(H) = [\Phi^{-1}(F) \cup \{0\}] \cap E(K).$$

Thus F consists of the sets $\{p, -p\}$ with $p \in E(H)$, and F is closed.

7. The Gleit-Taylor Theorem

Throughout this section, V will be a Lindenstrauss space, K the closed unit ball of V^*. The methods of this section are essentially due to Gleit and Taylor [13; 22].

The key result used by both Gleit and Taylor is the following partial converse to Proposition 4.7 (the full converse is false—see [22, §2.10]). We recall that a point p in a topological space is a *cluster point* of a net p_α if there is a subnet of p_α converging to p.

LEMMA 7.1. *Suppose that p_n is a* sequence *in $E(K)$ converging weak* to $q \in K$. Then the structure cluster points of p_n all lie in $E(H_q)$.*

PROOF. For each integer n, the set

$$D_n = \{\pm p_k : k \geqq n\} \cup H_q$$

is symmetrically dilated and weak*-closed. From Theorem 5.8 it follows that

$$D_n \cap E(K) = \{\pm p_k : k \geqq n\} \cup E(H_q)$$

is structurally closed. If p_α is a subnet of p_n that converges structurally to $p \in E(K)$, p_α must eventually lie in $D_n \cap E(K)$, hence $p \in D_n \cap E(K)$. Taking the intersection of these sets, $p \in E(H_q)$.

For each $p \in E(K)$ we define

$$\Gamma(p) = \{q \in Z : p \in E(H_q)\},$$
$$\Delta(p) = \cap\{\bar{N} : N \in \mathcal{N}_p\},$$

where $\mathcal{N}_p$ is the collection of structural neighborhoods of p in $E(K)$, and $\bar{N}$ is the weak* closure of N. If S is a subset of $E(K)$, we let $\Gamma(S) = \bigcup_{p \in S}\Gamma(p)$.

LEMMA 7.2. *$\Delta(p)$ consists of all $q \in K$ such that there exists a net $p_\alpha \in E(K)$ with $p_\alpha \to p$ structurally and $p_\alpha \to q$ weak*.*

PROOF. Suppose that $p_\alpha \to p$ structurally and $p_\alpha \to q$ weak*. If N is a structure neighborhood of p, eventually $p_\alpha \in N$, hence $q \in \bar{N}$, i.e., $q \in \Delta(p)$. If $q \in \Delta(p)$, let $\mathscr{G}_q$ be the neighborhoods of q in the weak* topology. Order $\mathscr{N}_p \times \mathscr{G}_q$ by $(N_1, G_1) \geqq (N_2, G_2)$ if $N_1 \subseteq N_2$ and $G_1 \subseteq G_2$. For each $N \in \mathscr{N}_p$, $G \in \mathscr{G}_q$ choose $p_{(N,G)} \in N \cap G$. Then $p_{(N,G)}$ is a net converging to p structurally and to q weak*.

COROLLARY 7.3. *For each $p \in E(K)$, $\Gamma(p) \subseteq \Delta(p)$.*

PROOF. This is immediate from Proposition 4.7.

COROLLARY 7.4. *If G is a weak* open set in K and $G \supseteq \Delta(p)$, $p \in E(K)$, then $G \cap E(K)$ is a structure neighborhood of p.*

PROOF. If $G \cap E(K)$ is not a structure neighborhood of p, let $p_\alpha \in E(K) - G$ converge structurally to p. Choosing a subnet we may assume that p_α converges weak* to $q \in K$. Then $q \in \Delta(p) \subseteq G$, hence eventually $p_\alpha \in G$, a contradiction.

LEMMA 7.5. *If V is separable and $D \subseteq E(K)$ is structure compact, then $\bar{D} \subseteq \Gamma(D)$.*

PROOF. If $q \in \bar{D}$, then since K is metrizable, there is a sequence $p_n \in E(K)$ converging weak* to q. Since D is structure compact, p_n has a cluster point p. From Lemma 7.1, $q \in \Gamma(p)$, hence $q \in \Gamma(D)$.

A topological space X is *locally compact* at p if each neighborhood of p contains a compact neighborhood of p.

THEOREM 7.6. *Suppose that V is a separable, Lindenstrauss space, K the closed unit ball of V^*, and p a point in $E(K)$. Then the following are equivalent:*

(1) $E(K)$ has a countable basis of structure open sets at p.

(2) $E(K)$ is structurally locally compact at p.

(3) $\Gamma(p) = \Delta(p)$.

PROOF. (1) $\Rightarrow$ (3). If $q \in \Delta(p)$, let p_α be a net in $E(K)$ with $p_\alpha \to p$ structurally, $p_\alpha \to q$ weak* (Lemma 7.2). Since by hypothesis there are countable bases in $E(K)$ and K at p and q for the structure and weak* topologies, respectively, we may select a sequence p_n from the p_α with $p_n \to p$ structurally and $p_n \to q$ weak*. From Lemma 7.1, $q \in \Gamma(p)$.

(2) $\Rightarrow$ (3). If V is locally compact, we have from Lemma 7.5

$$\begin{aligned}\Delta(p) &= \cap\ \{\bar{N}: N \in \mathcal{N}_p,\ N \text{ compact}\} \\ &\subseteq \cap\{\Gamma(N): N \in \mathcal{N}_p\} \\ &\subseteq \Gamma(p).\end{aligned}$$

To justify the last inclusion, suppose $q \notin \Gamma(p)$. Then $p \notin E(H_q)$, and letting $N = E(K) - E(H_q)$, $p' \notin E(H_q)$ for all $p' \in N$, i.e., $q \notin \Gamma(N)$.

(3) $\Rightarrow$ (1). Let G_n be a basis of weak* open sets for K, which is closed under finite unions. If N is a structure open set containing p, let $E(K) - N = E(H)$, H a closed biface. Since H is symmetrically dilated, $q \in H$ implies $E(H_q) \subseteq E(H)$, hence $q \notin \Gamma(p)$, i.e., $\Gamma(p) \cap H = \emptyset$. Thus (2) implies that $\Delta(p)$ and H are disjoint compact subsets of K. Thus there is a G_n with $\Delta(p) \subseteq G_n$, and $G_n \cap H = \emptyset$, and $G_n \cap E(K)$ is a structure neighborhood of p by Corollary 7.4.

(3) $\Rightarrow$ (2). Let N be a structure open set containing p, and H a closed biface with $E(K) - N = E(H)$. As in the proof of (3) $\Rightarrow$ (1), $\Delta(p) \cap H = \emptyset$. It suffices to construct a convex weak* continuous function f on K such that $f|H = 0$ and $f|\Delta(p) > \varepsilon$ for some $\varepsilon > 0$, since then the set

$$\{q \in E(K): f(q) \geqq \varepsilon\}$$

will be a compact neighborhood of p (Proposition 4.8 and Corollary 7.4.)

Since H is the unit ball of lin H, and H is weak* closed, the same is true for lin H (see [9, Ch. IV, §2, Th. 5].) Letting $W = V^*/\text{lin } H$ have the weak* quotient topology, the quotient map ϕ of V^* onto W is weak* continuous, and one has that the compact set $D = \phi(\Delta(p))$ does not contain 0. An obvious covering argument shows that there exist continuous linear functions $g_1, \cdots, g_n$ for which $g = g_1 \vee \cdots \vee g_n$ is larger than zero on D. Then $f = g \circ \phi$ has the desired properties.

In the proof of (3) $\Rightarrow$ (1), G_n was chosen to be a weak* basis for all of K. We may thus conclude.

THEOREM 7.7. *Suppose that V is a separable Lindenstrauss space, and K is the closed unit ball of V^*. Then the following are equivalent for the structure topology:*

(1) *$E(K)$ is first countable.*
(2) *$E(K)$ is second countable.*
(3) *$E(K)$ is locally compact.*

8. Further Remarks

8.1. A simple argument will give a representation theorem for a complex Banach space V. Let K be the closed unit ball of V, and let T be the complex numbers of modulus 1. Then V may be identified with $A_T(K,C)$, the continuous complex affine functions on K satisfying $f(\alpha p) = \alpha f(p)$ for all $\alpha \in T$. The Lazar criterion for Lindenstrauss-spaces has an obvious analogue. If f is a continuous complex function on K, define

$$\mathrm{odd}_T f(p) = \int \alpha f(\alpha^{-1} p) d\alpha$$

where integration is with respect to Haar measure on T. If $\mu \in P(K)$, define $\mathrm{odd}_T \mu$ by

$$\mathrm{odd}_T \mu(f) = \mu(\mathrm{odd}_T f).$$

We say that K is a *T-simplex*, and V is *T-simplicial*, if given maximal μ, $\nu \in P(K)$ and $r(\mu) = r(\nu)$, it follows that $\mathrm{odd}_T \mu = \mathrm{odd}_T \nu$. One finds that such a space V may be identified with $A_T(Z,C)$, the functions f whose real and imaginary parts lie in $A(Z)$, and satisfy $f(\alpha p) = \alpha f(p)$. Presumably much of the preceding theory will generalize if one appropriately defines "T-faces" in K.

These ideas might also be useful in the study of the action of a group G on a convex set K. If L is a representation of G on a Banach space W one might wish to consider the space $A_T(K, W)$ of affine functions $f: K \to W$ satisfying $f(\alpha p) = L(\alpha) f(p)$ for $\alpha \in G$.

8.2. The M-space version of Theorem 6.3 is valid in the non-separable case. This is due to a particularly simple proof for (3) $\Rightarrow$ (1) [12, Th .2.5]. A more thorough understanding of the operation $\curlywedge$ might lead to a corresponding result for G-spaces.

8.3. We conjecture that a Banach space V is a G-space if and only if a finite non-empty intersection of balls in V must be centrally symmetric. This is the case if V is a Lindenstrauss space.

8.4. If V is a C_σ-space, then given u, v, $w \in V$ there exists a unique element $uvw \in V$ satisfying

$$(uvw)(p) = u(p)v(p)w(p), \qquad p \in E(K) \tag{8.1}$$

Define a *C_σ-character* to be an element $p \in V^*$ satisfying (8.1). It would seem probable that the C_σ-characters are just the elements of $E(K) \cap \{0\}$, and that a Lindenstrauss space V is a C_σ-space if and only if $\overline{E(K)} \subseteq E(K) \cup \{0\}$.

8.5. We suspect that there exists a remnant of "M-structure" in V that would enable one to directly characterize the Lindenstrauss spaces, and discuss ideal theory in V. This would probably involve a Riesz decomposition property, that would be related to the "restricted four-two intersection property" (see [18, Th. 6.1]).

Notes added in proof (January 30, 1971).

1. Using rather different approaches, H. Fakhoury and P. Taylor have independently settled §8.2 by showing that the conditions (1)–(3) of Theorem 6.3 are equivalent without the assumption of separability.

2. H. Fakhoury has shown that the conjectures of §8.3 are true.

3. E. Alfsen and the author have recently succeeded in developing a structure theory for arbitrary non-ordered Banach spaces. In this context, we have found analogues of the Edwards-Lazar-Lindenstrauss Extension Theorem [17, Th. 2.1], the structure topology, and the Dauns-Hofmann Theorem (see Theorem 5.7). In addition, we have found an intrinsic characterization of the "ideals" in such spaces.

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